Homework 2 Solutions

1. True/False. If a statement is true, prove it. If it is false, provide a counter example.

   (a) If $AB = AC$, then $B = C$.

   False. Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $B \neq C$.

   (b) If $AB = AC$ and A is invertible, then $B = C$.

   True. Multiplying both sides of the equation by A^{-1} yields:

$$\begin{aligned} A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ IB &= IC \\ B &= C \end{aligned}$$

   (c) If a matrix has a column of zeroes, then it is not invertible.

   True. Let A be an $n \times n$ matrix whose i-th column, $\mathbf{a}_i$ is filled with zeros. Let B be any other $n \times n$ matrix. Then, by definition of matrix multiplication, the ith column of BA is the vector $B\mathbf{a}_i = \mathbf{0}$. Hence BA can never be the identity matrix, since its ith column is the zero vector. It follows that A has no inverse.

   (d) Any two elimination matrices (of the same size) commute with each other.

   False.

   Consider the 3×3 elimination matrices $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

   We have $E_{21}E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, whereas

   $E_{32}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

2. Show that the inverse of symmetric matrix is symmetric.

Let A be a symmetric matrix; then $A^T = A$. We need to show that A^{-1} is also its own transpose. By the transpose properties, $A^{-1} = (A^T)^{-1} = (A^{-1})^T$. Now since A^{-1} is its own transpose, A^{-1} is symmetric, by definition.

3. For what values of c is $A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$ invertible?

A is invertible if it has 3 pivots. To see the pivot positions, we can use Gaussian elimination.

$$\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \xrightarrow[\text{Replace R3 with R3-4R1}]{\text{Replace R2 with R2}-\frac{c}{2}\text{R1}} \begin{bmatrix} 2 & c & c \\ 0 & \frac{2c-c^2}{2} & \frac{2c-c^2}{2} \\ 0 & 7-4c & -3c \end{bmatrix}$$

$$\xrightarrow{\text{Replace R3 with R3}-\frac{14-8c}{2c-c^2}\text{R2}} \begin{bmatrix} 2 & c & c \\ 0 & \frac{2c-c^2}{2} & \frac{2c-c^2}{2} \\ 0 & 0 & c-7 \end{bmatrix}$$

The first row has a pivot, regardless of the value of c. The second row has a pivot only if $c \neq 0, 2$. The third row has a pivot only if $c \neq 7$. Hence the matrix is invertible for all $c \neq 0, 2, 7$.

4. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 2 \\ 6 & 4 & -1 \end{bmatrix}$.

   (a) Use Gauss-Jordan elimination to find the inverse of A.

$$\left[A \quad | \quad I\right] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 6 & 4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Replace R3 by R3-3R1}]{\text{Replace R2 by R2-R1}} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Replace R3 by R3+R2}} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Replace R2 by R2-2R3}} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 7 & -1 & -2 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Replace R1 by R1+R2}} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 8 & -1 & -2 \\ 0 & -1 & 0 & 7 & -1 & -2 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Multiply R2 by } -1]{\text{Multiply R1 by } \frac{1}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & -7 & 1 & 2 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 4 & -\frac{1}{2} & -1 \\ -7 & 1 & 2 \\ -4 & 1 & 1 \end{bmatrix}$

(b) Use the inverse matrix to solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Multiplying $A\mathbf{x} = \mathbf{b}$ on either side by A^{-1} yields:

$$\begin{aligned} A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

Hence $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 4 & -\frac{1}{2} & -1 \\ -7 & 1 & 2 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

5. Use the LU factorization

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 2 \\ 6 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

to solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

To solve $A\mathbf{x} = \mathbf{b}$, we first solve $L\mathbf{u} = \mathbf{b}$ using forward substitution and then solve $U\mathbf{x} = \mathbf{u}$ using backwards substitution.

$L\mathbf{u} = \mathbf{b}$ corresponds to the system

$$\begin{aligned} u_1 &= 1 \\ u_1 + u_2 &= 2 \\ 3u_1 - u_2 + u_3 &= 3 \end{aligned}$$

The first equation gives $u_1 = 1$; plugging into the second equation gives $u_2 = 1$; plugging into the third equation gives $u_3 = 1$. Hence $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now $U\mathbf{x} = \mathbf{u}$ corresponds to the system

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ -x_2 + 2x_3 &= 1 \\ x_3 &= 1 \end{aligned}$$

The last equation gives $x_3 = 1$; plugging into the second equation gives $x_2 = 1$; plugging into the first equation gives $x_1 = 0$. Hence the solution to the system is $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

6. Let A be a 3×3 matrix and consider the following elimination and permutation matrices.

$$E_{13} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Describe, in order, the row operations performed on A to obtain the matrix $E_{13}E_{21}P_{13}A$. We multiply A by P_{13}, then E_{21}, then E_{13}. In order, these multiplications correspond to:
  - Swapping R1 and R3;

– Replacing R2 with R2+2R1; and
– Replacing R1 with R1-R3.

- Let's verify your answer to part (a) in an example. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. First compute $E_{13}E_{21}P_{13}A$. Next, perform the row operations to A that you found in part (a). Finally, show that both computations yield the same matrix.

  First we compute the product:

$$\begin{aligned} E_{13}E_{21}P_{13}A &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 6 & 6 \\ 18 & 21 & 24 \\ 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

  Next we perform the for operations (in order):

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{\text{Swap R2 and R3}} \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{Replace R2 with R2+2R1}} \begin{bmatrix} 7 & 8 & 9 \\ 18 & 21 & 24 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{Replace R1 with R1-R3}} \begin{bmatrix} 6 & 6 & 6 \\ 18 & 21 & 24 \\ 1 & 2 & 3 \end{bmatrix}$$

  Notice that the two matrices are the same.

7. Let $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 2 & 4 & 2 & 1 \\ 6 & 3 & 2 & -2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$

(a) Use Gaussian elimination to put A in upper triangular form U and write down the corresponding elimination matrices. Express U as a product of A and the elimination matrices.

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 2 & 4 & 2 & 1 \\ 6 & 3 & 2 & -2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \xrightarrow[\text{by R3-3R1 and R4 by R4-R1}]{\text{Replace R2 by R2-R1, R3}} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{Replace R4 by R4-R3}} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The relevant elimination matrices (in order) are:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and $U = E_{43}E_{41}E_{31}E_{21}A$.

(b) Explain why the system $A\mathbf{x} = \mathbf{b}$ can not have a unique solution for all $\mathbf{b} \in \mathbb{R}^4$.

If the system had a unique solution, then we would A would have a pivot in every row. However, we see from U in part (a), that there is not a pivot in the final row. Hence there will never be a unique solution.

(c) Use the elimination matrices from part (a) to find an LU factorization of A.

$$L = (E_{43}E_{41}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{41}^{-1}E_{43}^{-1}$$
$$= \begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\3&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}$$
$$= \begin{bmatrix}1&0&0&0\\1&1&0&0\\3&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&1&1\end{bmatrix}$$
$$= \begin{bmatrix}1&0&0&0\\1&1&0&0\\3&0&1&0\\1&0&1&1\end{bmatrix}$$

(d) Let $\mathbf{b} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$. Find conditions on the entries of $\mathbf{b}$ so that the system $A\mathbf{x} = \mathbf{b}$ has a solution.

We'll use the LU factorization to solve the system $A\mathbf{x} = \mathbf{b}$ and consider the pivots to decide when a solution exists. We first solve $L\mathbf{u} = \mathbf{b}$ using forward substitution and then solve $U\mathbf{x} = \mathbf{u}$ using backwards substitution.

$L\mathbf{u} = \mathbf{b}$ corresponds to the system

$$\begin{aligned} u_1 &= b_1 \\ u_1 + u_2 &= b_2 \\ 3u_1 + u_3 &= b_3 \\ u_1 + u_3 + u_4 &= b_4 \end{aligned}$$

The first equation gives $u_1 = b_1$; plugging into the second equation gives $u_2 = b_2 - b_1$; plugging into the third equation gives $u_3 = b_3 - 3b_1$; plugging into the fourth equation gives $u_4 = 2b_1 - b_3 + b_4$. Hence $\mathbf{u} = \begin{bmatrix}b_1\\b_2 - b_1\\b_3 - 3b_1\\2b_1 - b_3 + b_4\end{bmatrix}$.

Now $U\mathbf{x} = \mathbf{u}$ corresponds to the system

$$\begin{aligned} 2x_1 + x_2 - x_4 &= b_1 \\ 3x_2 + 2x_3 + 2x_4 &= b_2 - b_1 \\ 2x_3 + x_4 &= b_3 - 3b_1 \\ 0 &= 2b_1 - b_3 + b_4 \end{aligned}$$

This system has a solution if and only if $2b_1 - b_3 + b_4 = 0$.

(e) Find a vector $\mathbf{b} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions.

Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. From part (c), we know that if $2b_1 - b_3 + b_4 \neq 0$, then there is no solution. So, for example, the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has no solution.